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PSEUDO JUMP OPERATORS. I: THE R. E. CASE

BY

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ABSTRACT. Call an operator J on the power set of ω a pseudo jump operator if $J(A)$ is uniformly recursively enumerable in A and A is recursive in $J(A)$ for all subsets A of ω. Thus the (Turing) jump operator is a pseudo jump operator, and any existence proof in the theory of r.e. degrees yields, when relativized, one or more pseudo jump operators. Extending well-known results about the jump, we show that for any pseudo jump operator J, every degree $\geqslant \mathbf{0}'$ has a representative in the range of J, and that there is a nonrecursive r.e. set A with $J(A)$ of degree $\mathbf{0}'$. The latter result yields a finite injury proof in two steps that there is an incomplete high r.e. degree, and by iteration analogous results for other levels of the H_n, L_n hierarchy of r.e. degrees. We also establish a result on pairs of pseudo jump operators. This is combined with Lachlan's result on the impossibility of combining splitting and density for r.e. degrees to yield a new proof of Harrington's result that $\mathbf{0}'$ does not split over all lower r.e. degrees.

1. We prove some generalizations of theorems about the Turing jump operator (denoted $A \mapsto A'$) to theorems about operators of the form $A \mapsto A \oplus W_e^A$, for an arbitrary fixed Gödel number e. (Here $A \oplus B$ is the recursive join of A and B and W_e^A is the eth set r.e. in A in a fixed standard enumeration.) For instance, Friedberg (see [**25**, Theorem 4.1]) showed that there is a nonrecursive r.e. set A with $A' \equiv_T K$ (where K is a complete r.e. set). We prove by a finite injury priority argument similar to Friedberg's that for every e there is a nonrecursive r.e. set A with $A \oplus W_e^A \equiv_T K$. Now Friedberg's argument relative to an arbitrary oracle B yields a fixed Gödel number i such that, for all B, W_i^B is low over B, i.e. $B <_T W_i^B$ and $B' \equiv_T (W_i^B)'$. Applying our result with $i = e$ we obtain a set A of incomplete high r.e. degree, i.e. $A <_T K$ and $A' \equiv_T K'$. Of course the existence of such a set has been known for some time [**20**], but this is apparently the first proof using only the finite injury method. (Thus a single infinite injury construction has been replaced by two successive applications of finite injury arguments.) An iteration of the uniform relative form of the argument shows in the same way that there are r.e. degrees at all levels of the high$_n$, low$_n$ hierarchy. Then an application of the recursion theorem yields a finite injury proof of the result of A. H. Lachlan [**10**] and D. A. Martin [**16**] that there is an r.e. degree $\mathbf{a}$ not high$_n$ or low$_n$ for any n (see also Sacks [**21**]). (A

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degree $\mathbf{a} \leqslant \mathbf{0}'$ is called high$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$ and low$_n$ if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$.) Of course our result may be applied to constructions other than that of a low nonzero r.e. degree, and we also draw conclusions from its application to the construction of a low nonbranching degree.

We also prove the following extension of Friedberg's jump theorem [**4**]: for every e and every set C with $K \leqslant_T C$, there is a set A with $A \oplus W_e^A \equiv_T A \oplus K \equiv_T C$. (The proof is the same as Friedberg's.)

Our strongest result is that for any r.e. set B of low degree and every pair of Gödel numbers e, i there is an r.e. set A such that $A \oplus W_e^A$ has r.e. degree, $A \oplus W_e^A \oplus W_i^A \equiv_T K$, and $B \leqslant_T A \oplus W_e^A$. Applied to Lachlan's result [**13**] on the impossibility of combining splitting and density, this yields Harrington's stronger result [**7**] that there is an r.e. degree $\mathbf{c} < \mathbf{0}'$ over which $\mathbf{0}'$ cannot be split by any pair of incomparable r.e. degrees, with the additional embellishment that $\mathbf{c}$ may be chosen to lie above any given low r.e. degree. Finally the two Gödel-number result above may be applied to Harrington's result [**6, 17**] that there is a high r.e. degree $\mathbf{c} < \mathbf{0}'$ which cannot be cupped up to $\mathbf{0}'$ by any incomplete r.e. degree. (This extended a result previously announced by C. E. M. Yates.) From the application it follows that there are r.e. degrees $\mathbf{h}, \mathbf{l}$ with $\mathbf{l}$ low, $\mathbf{h}$ high, $\mathbf{l} < \mathbf{h}$, such that every r.e. degree which cups $\mathbf{h}$ to $\mathbf{0}'$ also cups $\mathbf{l}$ to $\mathbf{0}'$, and $\mathbf{h}$ is cuppable to $\mathbf{0}'$ by a low r.e. degree.

The work presented here actually began with an attempt to distinguish $\mathbf{0}'$ from the other (r.e.) degrees by trying to find some special property that it alone might have inside every cone of degrees in which it lies. Thus we considered briefly whether within such cones $\mathbf{0}'$ might always (or never) be the sup of an r.e. minimal pair, or cuppable to everything above it, or branching, or nonbranching, etc. Some such questions could be answered from known results. For instance, the result of A. H. Lachlan [**14**] and R. I. Soare and J. R. Shoenfield (unpublished) that $\mathbf{0}'$ is the sup of an incomparable pair of r.e. degrees having an inf immediately implies that, relative to the inf, $\mathbf{0}'$ is the sup of an r.e. minimal pair. On the other hand, most such questions appeared to be very resistant to direct attack. Theorems 2.1 and 3.1 will imply that for any property $\mathcal{P}$ of r.e. degrees such that there is a uniformly relativizable recursive enumeration of a set whose degree has $\mathcal{P}$ (including all the examples just mentioned) there is a degree $\mathbf{a}$ relative to which $\mathbf{0}'$ has $\mathcal{P}$.

The Friedberg jump theorem may be iterated to show that every degree $\geqslant \mathbf{0}^{(n)}$ is the nth jump of some degree, and the corresponding result for transfinite iterations of the jump may be proved using forcing [**15**]. In a sequel [**8**] to the current paper, these results will be extended to show that if P is an operation which is a composition of n operations of the form $A \mapsto A \oplus W_e^A$, then every degree $\geqslant \mathbf{0}^{(n)}$ contains a set in the range of P. This result will also be extended to the transfinite, and the level ω case will be used in combination with the Ershov difference hierarchy and the Sacks minimal degree construction [**19**] to show that every degree $\geqslant \mathbf{0}^{(\omega)}$ is a minimal cover. (This extends the result of Harrington and Kechris [**5**] that every degree above that of Kleene's $\mathcal{O}$ is a minimal cover.) It follows, for instance, essentially as in Shore [**24**] that if $\mathbf{a}^{(n)} \geqslant \mathbf{0}^{(\omega)}$ for some n, then $\mathcal{D}(\geqslant \mathbf{a})$ (the partial ordering of degrees $\geqslant \mathbf{a}$) is not elementarily equivalent to the partial ordering of all degrees.

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2. We now prove our extended Friedberg completeness criterion.

THEOREM 2.1. *If* $0' \leq_T C$ *and* e *is any number, there is a set* A *such that* $A \oplus W_e^A \equiv_T A \oplus 0' \equiv_T C$.

PROOF. We obtain the characteristic function of A as the union of an increasing sequence of strings $\langle \sigma_n \rangle$, precisely as in the Friedberg jump theorem. Let σ_{-1} be the empty string.

Stage $2n$. If there exists a string $\sigma \supseteq \sigma_{2n}$ with $n \in W_e^\sigma$, let σ_{2n+1} be the least such σ. Otherwise let $\sigma_{2n+1} = \sigma_{2n}$. (Here $n \in W_e^\sigma$ means that $n \in W_e^A$ can be computed in at most $|\sigma|$ steps from the information $\sigma \subseteq A$. We are identifying A with its characteristic function and writing $|\sigma|$ for the length of the string σ.)

Stage $2n + 1$. Let $\sigma_{2n+2} = \sigma_{2n+1} \frown \langle C(n) \rangle$.

Since the even stages may be carried out recursively in $0'$ and the odd stages recursively in C, the sequence $\langle \sigma_n \rangle$ is recursive in C, so $A \leq_T C$. Also $n \in W_e^A$ iff $n \in W_e^{\sigma_{2n+1}}$, so $W_e^A \leq_T C$. It follows that $A \oplus W_e^A$, $A \oplus 0'$ are each recursive in C.

Note that $C(n) = \sigma_{2n+2}(|\sigma_{2n+2}| - 1)$, so C is recursive in the sequence $\langle \sigma_n \rangle$. Thus to show that C is recursive in each of $A \oplus W_e^A$, $A \oplus 0'$, it suffices to show that $\langle \sigma_n \rangle$ is recursive in each. But this is clear since the even stages can be carried out recursively in each of $0'$, W_e^A, and the odd stages may be carried out recursively in A.

This theorem has a number of easy consequences, such as the existence of an incomplete high degree below $0'$. However, we defer discussion of these until the next section where the case $0' \equiv_T C$ of Theorem 2.1 is improved by requiring A to be r.e. and nonrecursive.

It would be natural to conjecture that Theorem 2.1 would continue to hold if W_e^A were replaced by the eth set Σ_2^0 in A, denoted S_e^A, and $0'$ were replaced by $0''$. However this fails even for the (provably) Δ_2^0 in A sets as can be seen by choosing e so that S_e^A is uniformly Δ_2^0 in A and the degree of S_e^A is always minimal over that of A [**19**]. In this case the degree of $A \oplus S_e^A$ cannot be $\mathbf{0}^{(n)}$ for any $n < \omega$ by [**9**]. On the other hand, Harrington and Kechris [**5**] showed that, for any e with S_e^A uniformly Δ_2^0 in A and any degree $\mathbf{c}$ above that of Kleene's $\mathcal{O}$, there is a set A such that $A \oplus S_e^A$ has degree $\mathbf{c}$. Further results along this line will appear in [**8**].

3. We now prove our extension of Friedberg's result asserting the existence of a nonzero low r.e. degree.

THEOREM 3.1. *For any number* e *there is an r.e. nonrecursive set* A *such that* $A \oplus W_e^A \equiv_T 0'$.

PROOF. The negative aspect of the construction involves ensuring $W_e^A \leq_T 0'$. As in Soare's presentation [**25**, Theorem 4.1] of the Friedberg construction, this is accomplished by means of negative requirements N_n which attempt to preserve computations showing $n \in W_e^A$ by preventing numbers below the use of such computations from entering A. Let A^s be the set of numbers enumerated in A before stage s of the construction, and let $r(n, s)$ be the use function for computing $n \in W_e^{A^s}$, if $n \in W_{e,s}^{A^s}$, and $r(n, s) = 0$ otherwise.

The positive aspect of the construction involves ensuring that A is nonrecursive and that $K \leqslant_T A \oplus W_e^A$. For the former we have positive requirements P_{2n} that $W_n \neq \bar{A}$. For the latter we have a positive requirement P_{2n+1} asserting that if n is enumerated in K at s, then a "sufficiently small" number z_n is enumerated in A at stage s as a "trace". To keep the positive requirements from interfering with each other we insist that any number enumerated in A for the sake of P_n be an element of R_n (the nth row). (We assume the sets $R_0, R_1, \ldots$ are infinite, pairwise disjoint, and uniformly recursive.) In order to make it possible to bound the possible values of z_n from n with an oracle for $A \oplus W_e^A$ (to show $K \leqslant_T A \oplus W_e^A$) we always choose the element of R_n entering A for P_n to be as small as possible without injuring negative requirements N_m of higher priority, i.e. for $m \leqslant n$. For this reason we define $v(n, s)$ to be the least element z of R_n such that $z \geqslant r(m, s)$ for each $m \leqslant n$ and impose the condition

$$(*) \qquad A^{s+1} - A^s \subseteq \{v(n, s): n \in \omega\}.$$

(REMARK. This argument may well be visualized in terms of "movable markers" in which case $v(n, s)$ represents the position of the nth marker at the beginning of stage s.)

Fix a 1-1 recursive enumeration $\{k_s\}$ of K. The construction is as follows.

Stage s. By definition, P_{2n+1} requires attention at s iff $n = k_s$. Also P_{2n} requires attention at s iff $W_n^s \cap A^s = \varnothing$ and $v(2n, s) \in W_n^s$. Let

$$A^{s+1} = A^s \cup \{v(n, s): P_n \text{ requires attention at } s\}.$$

By convention, $W_n^s = \varnothing$ for $n \geqslant s$ so the sets A^s are finite and uniformly recursive. It follows that $A = \bigcup_s A^s$ is r.e.

Each positive requirement requires attention at most once. Given n, let s be a stage after which no P_m, $m < n$, requires attention. If $n \in W_e^{A^t}$ for some $t \geqslant s$, then no number below $r(n, t)$ enters A at or after stage t (because it could only enter for the sake of P_m, $m > n$, but would then have to respect the restraint $r(n, t)$). It follows in this case that $r(n, t) = r(n, u)$ for all $u \geqslant t$. If $n \in W_e^{A^t}$ holds for no $t \geqslant s$, then $r(n, t) = 0$ for $t \geqslant s$. In either case, $\lim_s r(n, s)$ exists, and $W_e^A \leqslant_T K$ by the limit lemma.

The above implies at once the $\lim_s v(n, s) = v(n)$ exists for each n. If $W_n \cap A = \varnothing$, the construction guarantees that $v(2n) \in A$ iff $v(2n) \in W_n$, so $\bar{A} \neq W_n$.

Observe that for each n, at most one element of R_n is in A since each positive requirement receives attention at most once. Let $f(n)$ be the canonical index of the finite set $A \cap R_n$. The constuction guarantees that $n \in K$ iff $A \cap R_n \neq \varnothing$, so K is recursive in f. To complete the proof it suffices to show that f is recursive in $A \oplus W_e^A$. Suppose inductively that $f(m)$ has been computed from $A \oplus W_e^A$ for all $m < n$. Let s be a stage such that $R_m \cap A^s = R_m \cap A$ for all $m < n$. Let $t \geqslant s$ be a stage such that $m \in W_e^{A^t}$ iff $m \in W_e^A$ holds for all $m \leqslant n$. Then, as in the argument that $\lim_s r(n, s)$ exists, it follows that $r(m, u) = r(m, t)$ for all $m \leqslant n$, $u \geqslant t$. Thus

$v(n, u) = v(n, t)$ for all $u \geq t$. If $v(n, t) \in A$, then $A \cap R_n = \{v(n, t)\}$. Otherwise $A \cap R_n = A^t \cap R_n$. In either case $f(n)$ has been computed from $A \oplus W_e^A$ and $\langle f(m): m < n \rangle$. This completes the proof.

As mentioned in the introduction, Theorem 3.1 and the existence of an e such that, for all A, $A <_T W_e^A$ and $A' \equiv_T (W_e^A)'$ immediately yields a finite injury proof (in two steps) of the existence of an incomplete high r.e. degree. To obtain further results along these lines, we apply the following corollary to the proof of Theorem 3.1. The corollary is obtained by noting that in the proof of Theorem 3.1 an index of the r.e. set A may be obtained uniformly from e, and that the entire result holds relative to an arbitrary oracle B. We write $J_e(B)$ for $B \oplus W_e^B$.

COROLLARY 3.2. *There is a recursive function f such that, for all B and all e,* $J_e(J_{f(e)}(B)) \equiv_T B'$ and $B <_T J_{f(e)}(B)$.

Let H_n, L_n denote the set of high$_n$, respectively low$_n$, degrees.

COROLLARY 3.3 (SACKS). *For each n there exist r.e. degrees in $H_{n+1} - H_n$ and $L_{n+1} - L_n$.*

PROOF. To have a strong inductive hypothesis, we prove the result in relativized form. Let L_n^B be the set of degrees $\mathbf{c}$ which are low$_n$ over deg(B), i.e. satisfy $\mathbf{b} \leq \mathbf{c} \leq \mathbf{b}'$ and $\mathbf{c}^{(n)} = \mathbf{b}^{(n)}$. Similarly, H_n^B is the set of degrees high$_n$ over deg(B). Choose e_0 so that $J_{e_0}(B) \equiv_T B'$ for all B. Let $e_1 = f(e_0)$, where f is as in Corollary 3.2. Then deg($J_{e_1}(B)$) is always in $L_1^B - L_0^B$. In general let $e_{n+1} = f(e_n)$. One easily sees by induction that deg($J_{e_{2n}}(B)$) is in $H_n^B - H_{n-1}^B$ and deg($J_{e_{2n+1}}(B)$) is in $L_{n+1}^B - L_n^B$. (The point is that deg(B') is in L_n^C iff deg(C) is in H_n^B, and deg(B') is in H_n^C iff deg(C) is in L_{n+1}^B, for $B \leq_T \mathbf{C} \leq_T B'$.)

COROLLARY 3.4 (MARTIN, LACHLAN, SACKS). *There is an r.e. degree not in H_n or L_n for any n.*

PROOF. Let f be as in Corollary 3.2. By the uniform relative version of the recursion theorem there is a number $\hat{e}$ such that $W_{f(\hat{e})}^B = W_{\hat{e}}^B$ (and hence $J_{f(\hat{e})}(B) = J_{\hat{e}}(B)$) for all B. Thus $B <_T J_{\hat{e}}(B)$ and $J_{\hat{e}}(J_{\hat{e}}(B)) \equiv_T B'$ for all B. (The operation $J_{\hat{e}}$ may be thought of as a sort of "half-jump" although we do not know whether it is degree invariant.) From the above it follows that $J_{\hat{e}}^{(2n)}(B) \equiv_T B^{(n)}$ for all B and n, and also that $J_{\hat{e}}^{(2n)}(B) <_T J_{\hat{e}}^{(2n+1)}(B) <_T J_{\hat{e}}^{(2n+2)}(B)$. The latter inequality may now be rewritten as $B^{(n)} <_T (J_{\hat{e}}(B))^{(n)} <_T B^{(n+1)}$, so $J_{\hat{e}}(B) \notin L_n^B \cup H_n^B$. Thus the degree of $J_{\hat{e}}(\varnothing)$ satisfies Corollary 3.4.

A form of the Sacks jump theorem asserts that if $\mathbf{c} \geq \mathbf{0}'$ is r.e. in $\mathbf{0}'$ then there is an r.e. degree $\mathbf{d}$ with $\mathbf{d}' = \mathbf{c}$. S. B. Cooper [**1**] thought that $\mathbf{d}$ could be chosen below any high degree. It would then follow that every degree $\mathbf{c} \geq \mathbf{0}'$ r.e. in $\mathbf{0}'$ is r.e. in every high degree. One of us (Shore) some time ago found a counterexample to a relativized version of the corollary. This relativized counterexample can now be transformed by our methods into an outright refutation of the claim. The proof uses the existence of a low r.e. degree which is not branching, which follows from the fact that every nonzero r.e. degree bounds a nonbranching r.e. degree [**23**, p. 106].

COROLLARY 3.5. *There is a degree* $\mathbf{c} \geqslant \mathbf{0}'$ *which is r.e. in* $\mathbf{0}'$ *and a high r.e. degree* $\mathbf{a}$ *such that* $\mathbf{c}$ *is not r.e. in* $\mathbf{a}$.

PROOF. Let e be a Gödel number such that, for every A, $\deg(W_e^A)$ is a low, nonbranching degree relative to $\deg(A)$. (Thus $A <_T W_e^A$, $A' \equiv_T (W_e^A)'$, and $\deg(W_e^A)$ cannot be expressed as the infimum (within the degrees r.e. in A) of an incomparable pair of degrees r.e. in A.) By Theorem 3.1, choose an r.e. set A with $W_e^A \equiv_T 0'$. Then A has high degree because $0'$ is low over A. By the minimal pair construction relative to $\mathbf{0}'$ [**11**], let $\mathbf{c}_0, \mathbf{c}_1$ be incomparable degrees r.e. in $\mathbf{0}'$ with infimum $\mathbf{0}'$ (among all degrees). For contradiction assume that both $\mathbf{c}_0, \mathbf{c}_1$ were r.e. in $\mathbf{a}$, where $\mathbf{a} = \deg(A)$. Then $\mathbf{0}'$ is branching over $\mathbf{a}$, contrary to choice of e. (One should note that if $\mathbf{c}_0, \mathbf{c}_1$ have inf $\mathbf{0}'$ among all degrees they have inf $\mathbf{0}'$ among degrees r.e. in $\mathbf{a}$.)

A much stronger refutation of Cooper's claim is provided by a recent result of Soare and Stob [**26**]: For every nonzero r.e. degree $\mathbf{c}$ there is a degree $\mathbf{d}$ which is r.e. in and above $\mathbf{c}$ but not r.e. (In fact by relativization this result implies that the degree $\mathbf{a}$ of Corollary 3.5 may be chosen to be any incomplete high degree below $\mathbf{0}'$.) It is worth noting that our results combined with a use of the recursion theorem show that one cannot get $\mathbf{d}$ uniformly in the Soare-Stob result, i.e. there is no recursive function g such that if W_e is nonrecursive then $J_{g(e)}(W_e)$ is not of r.e. degree. (Recall that $J_a(B)$ denotes $B \oplus W_a^B$.) To see this, assume g were such a recursive function and let f be a recursive function such that, for all a, $W_{f(a)}$ is nonrecursive and $J_a(W_{f(a)}) \equiv_T K$. (The existence of such an f follows from the uniformity of the proof of Theorem 3.1.) By the recursion theorem, there exists an e such that $W_e = W_{f(g(e))}$. We fix such an e and let $a = g(e)$. Then $W_{f(a)}$ is nonrecursive and $J_a(W_{f(a)}) \equiv_T K$. On the other hand, $W_{f(a)} = W_{f(g(e))} = W_e$, so $J_{g(e)}(W_e) = J_a(W_{f(a)})$ does not have r.e. degree by the assumed property of g. This is a contradiction.

It will be shown by a parallel argument in §4 that a weak uniformity in the Soare-Stob proof places a limitation on uniform extensions of our results. See also [**26**].

It follows from the Friedberg completeness criterion that $\mathbf{0}'$ has the cupping property, i.e. for any degree $\mathbf{c} \geqslant \mathbf{0}'$ there exists $\mathbf{b} < \mathbf{c}$ with $\mathbf{b} \cup \mathbf{0}' = \mathbf{c}$. The following corollary implies that it is not true that $\mathbf{0}'$ has the cupping property in all cones in which it lies.

COROLLARY 3.6. *There is an r.e. degree* $\mathbf{a} < \mathbf{0}'$ *and a degree* $\mathbf{c} > \mathbf{0}'$ *such that every degree* $\mathbf{b}$ *with* $\mathbf{a} \leqslant \mathbf{b} < \mathbf{c}$ *satisfies* $\mathbf{b} \leqslant \mathbf{0}'$.

PROOF. A result of S. B. Cooper [**2**] asserts that there is a nonzero r.e. degree $\mathbf{d}$ which has a strong minimal cover $\mathbf{e}$ (i.e. $\mathbf{d} < \mathbf{e}$ and every degree $< \mathbf{e}$ is $\leqslant \mathbf{d}$). By relativizing his result and applying Theorem 3.1 there is an r.e. degree $\mathbf{a} < \mathbf{0}'$ relative to which $\mathbf{0}'$ has a strong minimal cover $\mathbf{c}$.

4. We now obtain a generalization of Theorem 3.1 which applies to pairs of Gödel numbers. This will then be combined with constructions of comparable pairs of r.e. degrees, such as that of Lachlan's "monster paper" [**13**].

THEOREM 4.1. *Let e, i be given Gödel numbers, and let B be a given r.e. set of low degree. Then there is an r.e. set A such that $B \leq_T A \oplus W_e^A$, $A \oplus W_e^A$ is of r.e. degree, and $A \oplus W_e^A \oplus W_i^A \equiv_T K$.*

PROOF. We ensure that $A \oplus W_e^A$ is of r.e. degree using the same negative requirements N_n as in Theorem 3.1. Specifically, let $r(n, s)$, $v(n, s)$ be defined from the use of the computation (if any) showing $n \in W_{e,s}^{A^s}$ as in the proof of Theorem 3.1. We will continue to observe the restriction (*) from Theorem 3.1 that $A^{s+1} - A^s$ may contain only numbers of the form $v(n, s)$. Also it will continue to be true that $|A \cap R_n| \leq 1$ for each n. These restrictions are sufficient to ensure, as in Theorem 3.1, that f is recursive in $A \oplus W_e^A$, where $f(n)$ is the canonical index of the finite set $A \cap R_n$. From this we conclude, as in the proof of Theorem 3.1, that there is a function $g \leq_T A \oplus W_e^A$ such that, for all $s \geq g(n)$, $n \in W_e^A$ iff $n \in W_{e,s}^{A^s}$. (Let $g(n)$ be the least s such that $R_m \cap A^s = R_m \cap A$ for all $m \leq n$ and $n \in W_{e,s}^{A^s}$ iff $n \in W_e^A$.) Thus the modulus of convergence of the recursive approximation $A^s \oplus W_{e,s}^{A^s}$ to $A \oplus W_e^A$ may be computed recursively in $A \oplus W_e^A$, so $A \oplus W_e^A$ has r.e. degree by [**22**, Theorem 2].

Let P_n be the positive requirement which asserts that $n \in B$ iff $R_{2n} \cap A \neq \varnothing$. By the existence of f as above, the satisfaction of the P_n's implies that B is recursive in $A \oplus W_e^A$.

To ensure that $W_i^A \leq_T K$ we have negative requirements $\hat{N}_n$ which attempt to preserve computations of the form $n \in W_{i,s}^{A^s}$. Let $\hat{r}(n, s)$ be the corresponding restraint function (to be defined more precisely later). The negative requirements $\hat{N}_n$ will not exercise any restraint on the positive requirements P_m (but will restrain actions for other positive requirements $\hat{P}_m$ to be discussed). Thus $\hat{N}_n$ may be injured infinitely often by various P_m's and $\lim_s \hat{r}(n, s)$ may not exist. However the set of numbers put into A by all the requirements P_m will be recursive in B, and this will suffice to show that $W_i^A \leq_T B' \leq_T K$.

In order to ensure $K \leq_T A \oplus W_e^A \oplus W_i^A$ we impose positive requirements $\hat{P}_n$. If n is in K, according to $\hat{P}_n$ we must enumerate one or more "appropriate traces" into A. The requirement $\hat{P}_n$ is subject to the restraints imposed by $\hat{N}_m$ for $m \leq n$ and also to the condition (*). To ensure the existence of available traces of the form $v(k, s)$ which exceed $\hat{r}(m, s)$ for $m \leq n$, infinitely many rows R_k are set aside for $\hat{P}_n$ (specifically all rows R_{2k+1} with $k \in R_n$). The requirement $\hat{P}_n$ then asserts that $n \in K$ iff $R_{2k+1} \cap A \neq \varnothing$ for some $k \in R_n$. The satisfaction of these requirements does not itself guarantee that $K \leq_T A \oplus W_e^A \oplus W_i^A$, but in fact it will be possible to bound this quantifier on k recursively in $A \oplus W_i^A$. (This suffices since $\{k: R_k \cap A \neq \varnothing\}$ is recursive in $A \oplus W_e^A$ as remarked.) For this bounding on k to be possible it is important that $\liminf_s \hat{R}(n, s)$ exist, where $\hat{R}(n, s)$ is defined to be $\max\{\hat{r}(m, s): m \leq n\}$. For this reason, we use the Lachlan-Soare "hat trick" [**25**] and define $\hat{r}(n, s)$ to be the use function of the computation $n \in W_{i,s}^{A^s}$ if n belongs to both $W_{i,s}^{A^s}$ and $W_{i,s+1}^{A^{s+1}}$ by the same computation and $\hat{r}(n, s) = 0$ otherwise. Also let $k(n, s)$ be the least number $k \in R_n$ with $v(2k + 1, s) \geq \hat{R}(n, s)$. (Such a k exists because R_n is infinite and $v(k, s) \in R_k$ is 1-1 as a function of k.)

Fix recursive enumerations $\{K^s\}, \{B^s\}$ of K, B respectively with $B^0 = \varnothing$. The construction is as follows.

Stage s. For each $n \in B^{s+1} - B^s$, enumerate $v(2n, s)$ into A and say that P_n receives attention. For each $n \in K^s$, enumerate $v(2k(n, s) + 1, s)$ into A (and say that $\hat{P}_n$ receives attention at s) unless $R_{2j+1} \cap A^s \neq \varnothing$ for some $j \in R_n$ with $j \leqslant k(n, s)$.

This construction obviously satisfies $(*)$ from Theorem 3.1 and ensures that $|A \cap R_n| \leqslant 1$ and P_n is satisfied for all n. Thus, by the remarks before the construction, $A \oplus W_e^A$ has r.e. degree and $B \leqslant_T A \oplus W_e^A$. It remains to show that $A \oplus W_e^A \oplus W_i^A \equiv_T K$.

We establish first that $W_i^A \leqslant_T B'$, from which it follows that $W_i^A \leqslant_T K$. (This is the only use of the assumption that deg(B) is low.) Note that each requirement $\hat{P}_m$ receives attention only finitely often. (If $\hat{P}_m$ receives attention at stages $s_0 < s_1 < \cdots$, then $k(n, s_0) > k(n, s_1) > \cdots$ by construction.) A convergent computation $n \in W_{i,s}^{A^s}$ is called B-correct if for every m such that R_{2m} has a member $< \hat{r}(n, s)$, the equivalence $m \in B$ iff $R_{2m} \cap A^s \neq \varnothing$ holds. (This just says that the computation will not be destroyed by the action of any requirement P_m.) Given n, we now determine whether $n \in W_i^A$ recursively from B'. First let $s(n)$ be a stage such that no $\hat{P}_m$ with $m < n$ requires attention at any stage $s \geqslant s(n)$. This stage exists because each $\hat{P}_m$ requires attention only finitely often and it may be computed effectively from K, hence from B'. We claim now that $n \in W_i^A$ iff there is a B-correct computation $n \in W_{i,s}^{A^s}$ for some $s \geqslant s(n)$. Since "B-correctness" is recursive in B (as a predicate of n and s), the claim gives a method effective in B' for determining whether $n \in W_i^A$. The "only if" part of the claim is obvious. Suppose now that there is a B-correct computation $n \in W_{i,s}^{A^s}$ at stage $s \geqslant s(n)$. Then one easily shows by induction on t that no number below $\hat{r}(n, s)$ enters A at any stage $t \geqslant s$, so $n \in W_i^A$. (Briefly, the requirements P_m cannot cause such a number to enter A by B-correctness, the requirements $\hat{P}_m$ for $m < n$ also cannot because of the choice of $s(n)$, and the requirements $\hat{P}_m$ for $m \geqslant n$ must respect the higher priority restraint given by $\hat{r}(n, s) = \hat{r}(n, t)$.) This completes the proof that $W_i^A \leqslant_T K$. Also $A \oplus W_e^A \leqslant_T K$ since $A \oplus W_e^A$ has r.e. degree. Therefore $A \oplus W_e^A \oplus W_i^A \leqslant_T K$.

It remains only to show that $K \leqslant_T A \oplus W_e^A \oplus W_i^A$. It is clear from the construction that each $\hat{P}_n$ is satisfied, i.e. $n \in K$ iff $R_{2k+1} \cap A \neq \varnothing$ for some k in R_n. Thus, as remarked before the construction, it suffices to produce a function b, recursive in $A \oplus W_i^A$, such that, for all n, if $n \in K$ then $R_{2k+1} \cap A \neq \varnothing$ for some k in R_n with $k \leqslant b(n)$. Following **[25]**, we call a stage s *true* if every number enumerated in A at any stage $t > s$ exceeds some number enumerated in A at stage s. (We may assume without loss of generality that B is infinite and $B^{s+1} - B^s \neq \varnothing$ for all s, so that $A^{s+1} - A^s \neq \varnothing$ for all s.) Given n, let $t(n)$ be a true stage s such that, for each $m \in W_i^A$ with $m \leqslant n$, m is in both $W_{i,s}^{A^s}$ and $W_{i,s+1}^{A^{s+1}}$ via the same computation. Then t is a total function because the set T of true stages is infinite, and t is recursive in $A \oplus W_i^A$ because T is recursive in A. Since computations which survive true stages are permanent **[25**, (2.3)**]**, $\hat{R}(n, s) = \hat{R}(n, t(n))$ whenever s is a true stage $\geqslant t(n)$. Let $b(n)$ be the least number k in R_n such that $\min(R_{2k+1}) \geqslant \hat{R}(n, t(n))$. Then

$b \leqslant_T t \leqslant_T A \oplus W_i^A$ and $b(n) \geqslant k(n, s)$ whenever s is a true stage $\geqslant t(n)$. (The latter follows from the definition of $k(n, s)$ and the inequalities $v(2b(n)+1, s) \geqslant \min(R_{2b(n)+1}) \geqslant \hat{R}(n, t(n)) = \hat{R}(n, s)$.) Now suppose $n \in K$ and s is a true stage $\geqslant t(n)$ with $n \in K^s$. Then $k(n, s) \leqslant b(n)$ and so by construction either $R_{2j+1} \cap A^s \neq \varnothing$ for some $j \in R_n$ with $j \leqslant k(n, s)$ or $v(2k(n, s)+1, s)$ is enumerated in A at s. In either case $A \cap R_{2k+1} \neq \varnothing$ holds for some k in R_n with $k \leqslant b(n)$, so the proof is complete.

The above proof actually shows that for any numbers e, i and any r.e. set B (not necessarily of low degree), there is an r.e. set A such that $A \oplus W_e^A$ has r.e. degree, $B \leqslant_T A \oplus W_e^A$, and $K \leqslant_T A \oplus W_e^A \oplus W_i^A \leqslant_T B'$. However it is not known whether the condition $B \leqslant_T A \oplus W_e^A$ can be strengthened to $B \leqslant_T A$ even with the hypothesis that B is low.

We mention in passing some alternate proofs of Theorem 4.1. The proof just given could of course be done with movable markers. There would be two infinite sets of movable markers, the "e-markers" and the "i-markers". The nth e-marker moves so that its position exceeds the use function for $n \in W_{e,s}^{A^s}$ and the position of the $(n-1)$st e-marker. The i-markers behave analogously for $W_{i,s}^{A^s}$ except that the i-markers must move within the e-marker positions. The e-marker positions are used to code B and the i-marker positions are used to code K. Another approach eliminates the "infinite injury" aspects of the argument in favor of a stronger use of the lowness of deg(B). Specifically the question of whether there is a B-correct computation $n \in W_{i,s}^{A^s}$ for some $s \geqslant t$, for given n and t, is recursive in B' and hence recursively approximable. The recursion theorem allows recursive approximations to the answer to be used during the construction and thus ensure convergence with s of a modified $\hat{r}(n, s)$. This is known as the Robinson trick (see **[18]**). When carried out in detail it is a bit more complicated than the argument presented here and does not yield the strong form of the theorem for arbitrary r.e. B mentioned in the last paragraph.

The following corollary is just the special case of Theorem 4.1 obtained by choosing e so that $W_e^A = \varnothing$ for all A.

COROLLARY 4.2. *For any Gödel number i and low r.e. set B, there is an r.e. set A with $B \leqslant_T A$ and $A \oplus W_i^A \equiv_T 0'$.*

Using Corollary 4.2 (appropriately relativized) in place of Theorem 3.1 one may readily see that the r.e. degrees asserted to exist in Corollaries 3.3–3.5 exist above any given low r.e. degree.

A. H. Lachlan showed **[13]** that there are r.e. degrees $\mathbf{a}, \mathbf{c}$ such that $\mathbf{a} > \mathbf{c}$ and $\mathbf{a}$ cannot be split over $\mathbf{c}$ by r.e. degrees (i.e. there do not exist incomparable r.e. degrees $\mathbf{d}_0, \mathbf{d}_1 \geqslant \mathbf{c}$ with $\mathbf{d}_0 \cup \mathbf{d}_1 = \mathbf{a}$.) L. Harrington **[7]** then showed that one may choose $\mathbf{a} = \mathbf{0}'$ in Lachlan's theorem. Here we use Lachlan's result (in relativized form) to give a simpler proof of Harrington's generalization, with the slight further improvement that $\mathbf{c}$ may be chosen above any given low r.e. degree.

COROLLARY 4.3. *If $\mathbf{b}$ is any low r.e. degree, there is an r.e. degree $\mathbf{c} \geqslant \mathbf{b}$ such that $\mathbf{c} < \mathbf{0}'$ and $\mathbf{0}'$ cannot be split over $\mathbf{c}$ by r.e. degrees.*

PROOF. By the relativized form of Lachlan's theorem [**13**], there exist numbers e and i such that, for all A, $A <_T W_e^A <_T W_i^A$ and $\deg(W_i^A)$ cannot be split over $\deg(W_e^A)$ by degrees r.e. in A. Apply Theorem 4.1 to e, i and B, where B is an r.e. set of degree $\mathbf{b}$. Let $\mathbf{c} = \deg(W_e^A)$, where A is from Theorem 4.1.

Lachlan [**12**] defined an r.e. set A to be a major subset of an r.e. set B if $A \subseteq B$, $B - A$ is infinite, and $A \cup C$ is cofinite for every r.e. set C with $B \cup C$ cofinite. Also he has informally suggested the study of the corresponding concept for r.e. degrees. Specifically, if $\mathbf{a}, \mathbf{b}$ are r.e. degrees, we call $\mathbf{a}$ a major subdegree of $\mathbf{b}$ if $\mathbf{a} < \mathbf{b}$ and $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'$ for every r.e. degree $\mathbf{c}$ with $\mathbf{b} \cup \mathbf{c} = \mathbf{0}'$. An r.e. degree $\mathbf{b}$ is called cuppable if there is an r.e. degree $\mathbf{c} < \mathbf{0}'$ with $\mathbf{b} \cup \mathbf{c} = \mathbf{0}'$. The existence of a noncuppable, nonzero r.e. degree was announced by C. E. M. Yates, and proofs of this result and extensions thereof have been given by S. B. Cooper [**3**], L. Harrington [**6**], and D. Miller [**17**]. Of course, if $\mathbf{b}$ is a noncuppable r.e. degree, then every r.e. degree $\mathbf{a} < \mathbf{b}$ is a major subdegree of $\mathbf{b}$. The following corollary shows that it is also possible for a cuppable degree to have a major subdegree. It is not known whether every nonzero r.e. degree has a major subdegree.

COROLLARY 4.4. *There is a cuppable high degree* $\mathbf{h}$ *and a low degree* $\mathbf{l}$ *such that* $\mathbf{l}$ *is a major subdegree of* $\mathbf{h}$.

PROOF. Harrington [**6**] has shown the existence of a high, noncuppable r.e. degree. A proof is also given by Miller in [**17**]. Relativizing this result yields a number e such that for every A, $A <_T W_e^A$, $(W_e^A)' \equiv_T A''$, and $\deg(W_e^A)$ cannot be cupped to $\deg(A')$ by any degree above $\deg(A)$ and r.e. in A. Let B be an r.e. set such that $\deg(B)$ is low and cuppable. (The Sacks splitting theorem yields low r.e. degrees $\mathbf{b}_0, \mathbf{b}_1$ with $\mathbf{b}_0 \cup \mathbf{b}_1 = \mathbf{0}'$.) Let i be a number such that $W_i^A \equiv_T A'$ for all A. Apply Theorem 4.1 to obtain an r.e. set A with $A \oplus W_e^A$ of r.e. degree, $B \leqslant_T A \oplus W_e^A$, and $A \oplus W_e^A \oplus W_i^A \equiv_T 0'$. Let $\mathbf{l} = \deg(A)$, $\mathbf{h} = \deg(A \oplus W_e^A) = \deg(W_e^A)$. Clearly $\mathbf{l} < \mathbf{h}$, and $\mathbf{h}$ is cuppable because $\deg(B)$ is cuppable. (Indeed $\mathbf{h}$ is cuppable by a low degree if B is chosen as indicated above.) Also $A' \equiv_T A \oplus W_e^A \oplus W_i^A \equiv_T 0'$ so $\mathbf{l}$ is low. In addition $(W_e^A)' \equiv_T A'' \equiv_T 0''$ so $\mathbf{h}$ is high. Finally suppose $\mathbf{c}$ is an r.e. degree with $\mathbf{h} \cup \mathbf{c} = \mathbf{0}'$ in order to prove $\mathbf{l} \cup \mathbf{c} = \mathbf{0}'$. Then $\mathbf{h} \cup (\mathbf{l} \cup \mathbf{c}) = \mathbf{0}'$, so $\mathbf{l} \cup \mathbf{c}$ is a degree which is r.e. in $\mathbf{l}$, $\geqslant \mathbf{l}$, and cups $\deg(W_e^A)$ to $\mathbf{0}'$ ($= \deg A \oplus W_e^A \oplus W_i^A$). Therefore $\mathbf{l} \cup \mathbf{c} = \mathbf{l}' = \mathbf{0}'$ by choice of e and i.

In Theorem 4.1 it would have been a routine matter to insert additional requirements $W_n \neq \bar{A}$ to ensure that the constructed set A is nonrecursive. It thus follows that for every pair e, i of Gödel numbers there is a nonrecursive r.e. set A such that $A \oplus W_e^A$ and $A \oplus W_e^A \oplus W_i^A$ each have r.e. degree. It is then natural to ask whether for every pair e, i there is a nonrecursive r.e. set A such that $A \oplus W_e^A$, $A \oplus W_i^A$ each have r.e. degree. (A positive solution would allow the existence of a nonzero branching degree to be deduced from the construction of an r.e. minimal pair.) Although the question for two Gödel numbers just raised is still open, Soare and Stob have pointed out that, in contrast to the case where $J_e(A) \leqslant_T J_i(A)$, a positive answer may not be obtained by a uniform construction (see [**26**]). More precisely, they have shown that there is no binary recursive function f such that, for all e and i, $W_{f(e,i)}$ is nonrecursive and $J_e(W_{f(e,i)})$, $J_i(W_{f(e,i)})$ each have r.e. degree. (Here, as in

§3, $J_e(B)$ denotes $B \oplus W_e^B$.) (The proof uses the fact that the proof of the main result of **[26]** is semiuniform in the sense that there are recursive functions g_1, g_2 such that, for every e, if W_e is nonrecursive then either $g_1(e)$ or $g_2(e)$ is an index k such that $J_k(W_e)$ is not of r.e. degree. Given f as above, the recursion theorem yields an index j such that $W_j = W_{f(g_1(j), g_2(j))}$. It follows that f fails to have its assumed property when $e = g_1(j)$, $i = g_2(j)$. This argument is precisely analogous to the proof given in §3 that the Soare-Stob result cannot hold uniformly.) It seems conceivable that the methods of Soare and Stob may eventually lead to a negative answer to the question raised above.

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